The horizon scanner measurement relates to the local vertical only and does not contain any information concerning the yaw about the local vertical. This is reflected in the absence of ψ in the measurement Eq. (4). Equation (4) alone, of course, does not determine the direction of the local vertical completely. However, when the Earth is in view, for each revolution of the scanner there are two measurement equations like Eq. (4), corresponding to the horizon entry and exit. For fast scanning, the change in the spacecraft attitude and orbit during the short time interval it takes the scanning vector to sweep across the horizon circle may be neglected. The measurements at horizon entry and exit gives two components of the direction of the local vertical. From this information the roll ϕ of the spacecraft may be determined, but there still leaves a two-fold ambiguity about the pitch θ. Furthermore, based on the fast scanning assumption, measurements at successive horizon transits also convey information about the spacecraft attitude rate. To prove these assertions one may write down another measurement equation for later horizon transit similar to Eq. (4) as

$$-\left[1-\left(\frac{r}{r+h+\Delta h}\right)^2\right]^{1/2} = -\cos a\sin(\theta+\Delta\theta)$$
$$+\sin a\cos(\theta+\Delta\theta)\cos(\phi+\Delta\phi+\lambda+\Delta\lambda) \qquad (6)$$

where Δ represent the increments in the respective quantities since the horizon transit described by Eq. (4).

In most applications the scanning speed is fast. As a first approximation one may assume that neither the spacecraft altitude nor the attitude has changed; i.e., $\Delta h = \Delta\theta = \Delta\phi = 0$. One obtains immediately from Eqs. (4) and (6) that $\cos(\phi+\lambda) = \cos(\phi+\lambda+\Delta\lambda)$, or, the roll as

$$\phi_0 = (\lambda + \tfrac{1}{2}\Delta\lambda) \qquad (7)$$

The pitch then follows from Eqs. (5) and (7) as

$$\theta_0 = \mu_0 + \alpha_0, \text{ or } \pi - \mu_0 + \alpha_0 \qquad (8)$$

where

$$\mu_0 = \sin^{-1}\left\{\frac{1-[r/(r+h)]^2}{1-\sin^2 a\sin^2\frac{1}{2}\Delta\lambda}\right\}^{1/2}$$

$$\alpha_0 = \tan^{-1}\{\tan a\cos\tfrac{1}{2}\Delta\lambda\}$$

Usually Eqs. (4) and (6) represent measurements at horizon entry and exit, respectively and $\Delta\lambda$ becomes the "earth width" as seen by the scanner. The two possible pitch attitudes as given in Eq. (8) are illustrated in Fig. 3. The errors committed in neglecting the attitude changes $\Delta\theta$ and $\Delta\phi$ may be obtained from Eqs. (4) and (6) by a perturbation analysis as

$$e_\phi = \phi - \phi_0 \cong \frac{-\Delta\theta}{2\sin\frac{1}{2}\Delta\lambda}(\cot a + \tan\theta_0\cos\tfrac{1}{2}\Delta\lambda) - \tfrac{1}{2}\Delta\phi \qquad (9)$$

$$e_\theta = \theta - \theta_0 \cong e_\mu + e_\alpha, \text{ or } -e_\mu + e_\alpha \qquad (10)$$

where

$$e_\mu = -e_\phi \frac{\tan\mu_0}{(1-\sin^2 a\sin^2\frac{1}{2}\Delta\lambda)}(\sin^2 a \sin\tfrac{1}{2}\Delta\lambda) \qquad (11)$$

$$e_\alpha = e_\phi \frac{\tan a\sin\frac{1}{2}\Delta\lambda}{1+\tan^2 a\cos^2\frac{1}{2}\Delta\lambda} \qquad (12)$$

The small attitude changes $\Delta\theta$ and $\Delta\phi$ are of course related in the usual way to the angular velocity vector of the Nonscanning Axes relative to the Orbital Axes. Generally the attitude time constant is much shorter than the orbital time constant and the error due to the change in spacecraft altitude is extremely small in almost all cases. Equations (9-12) also serve as measurement equations for a second approximation where small attitude changes are considered. These equations should be valid for several revolutions of the scanner and may be related to the attitude rates as well as the attitude.

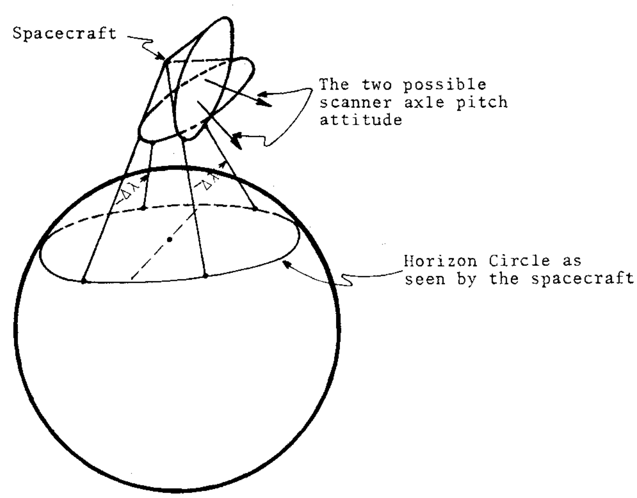

Fig. 3 The same Earth width measurement $\Delta\lambda$ corresponds to two possible pitch attitudes.

Sometimes two scanners with different half-cone angles are mounted on the same axle. Based on the fast scanning approximation then for each revolution of the scanner, four measurements about the local vertical are made. Not only the pitch may be determined now without ambiguity, but there are also redundant information for data smoothing. In fact, one may obtain the pitch and roll from Eqs. (4) and (6) as

$$\theta_0 = \tan^{-1}\left[\frac{\sin a'\cos\frac{1}{2}\Delta\lambda' - \sin a''\cos\frac{1}{2}\Delta\lambda''}{\cos a' - \cos a''}\right] \qquad (13)$$

$$\phi_0 = \lambda' + \tfrac{1}{2}\Delta\lambda' = \lambda'' + \tfrac{1}{2}\Delta\lambda'' \qquad (14)$$

where the superposed $'$ and $''$ refer to the two scanners. Equation (13) shows the redundancy eliminates the necessity of knowing the orbital information. Equation (14) indicates the roll may now be determined with greater accuracy.

So far the measurements are expressed in terms of scanning angles. Generally, actual measurements are horizon transit times. The transit time and the scanning angle are related by the scanning rate and any small rate biases will amplify with time. This means the roll error may become large if some means of periodic reinitialization is not provided. On the other hand, the pitch is related to the Earth width $\Delta\lambda$, which is sensitive to triggering biases but not to rate biases.

Reference

[1] Hatcher, N.M., "A Survey of Attitude Sensors for Spacecraft," NASA SP-145, 1967.

More on the Plane Turbulent Jet

Knox Millsaps*
University of Florida, Gainesville, Fla.

Nomenclature†

c = a scaling factor
f,p,q,r,s,t = mathematical functions occurring in the analysis
ℓ = Prandtl mixing length (ℓ)

Received July 18, 1975; revision received August 15, 1975.
Index categories: Jets, Wakes, and Viscid-Inviscid Flow Interactions; Nozzle and Channel Flow; Viscous Nonboundary-Layer Flows.

*Professor and Chairman, Department of Engineering Sciences. Associate Fellow AIAA.

†(m, ℓ and t express dimensions of physically significant quantities.)

u,v	= components of the turbulent velocity (ℓ/t)
x,y	= rectangular Cartesian coordinates (ℓ)
α	= a Galerkin constant
β,δ,m	= mathematical exponents
γ	= particular value of variable ξ denoting the termination of the jet
ξ,η	= similar variables
ρ	= density of incompressible fluid (m/ℓ^3)
ψ	= Stokes stream function (ℓ^2/t)

IN 1926, W. Tollmien[1] applied the Prandtl mixing length of turbulence to the submerged plane turbulent jet and obtained a now classical solution to the resulting mathematical problem for which an excellent review is given by Abramovich.[2] In the formulation Tollmien connected the changes in turbulent properties with the geometry of the jet by assuming a linear variation between the mixing length and the width of the jet or, equivalently, with the distance from the point of origin of the jet. With only one adjustable constant there is a remarkable correlation between experimental data and the analytically predicted velocity profiles although the curvature of the horizontal turbulent velocity profile is too large near the plane of symmetry of the jet and although there are obviously systematic errors in the tests for the assumed direct reciprocity in the similar variable y/x.

Although the Tollmien work has been modified and significantly improved upon by many researchers, it is still pedagogically pertinent as an introduction to the study of turbulent jets, and it may thus be worthwhile to note that a simple variation, apparently unnoticed, can be made which improves the agreement with measurements with respect to either similarity or shape. Also the minor improvement by an experimentally determined similarity transportation may be inherently interesting due to its possible use in other problems.

If one assumes a similar stream function $\psi = x^\delta f(yx^\beta) = x^\delta f(\eta)$ is the solution to the plane turbulent jet (the present notation is almost universally used), then the requirement for conservation of the x-component of momentum leads to the constraint

$$2\rho\int_0^\infty u^2 \mathrm{d}y = 2\rho\int_0^\infty x^{2(\delta+\beta)}[f'(\eta)]^2\mathrm{d}y$$

$$= 2\rho\int_0^\infty x^{2\delta+\beta}(f')^2\mathrm{d}\eta = \text{const}$$

and, if one is to have similarity, then $2\delta+\beta=0$. Also, if the Prandtl mixing length equation

$$u(\partial u/\partial x) + v\partial u/\partial y = 2\ell^2(\partial u/\partial y)/(\partial^2 u/\partial y^2$$

where one now assumes $\ell = cx^m$ in which $m>0$ is an exponent to be determined later, is to have a similar solution in yx^β, then one must require that $2\beta - 1 = 5\beta + 2m$ or $\beta = (-1-2m)/3$. Tollmien treated the case $m=1$ when the similar variable is y/x.

The partial differential equation can be reduced by the similar transformation to an ordinary differential equation

$$(f')^2 + ff' = 2f''f'''$$

where the primes denote differentiation with respect to ξ and where $\xi = [6c^2/(1+2m)]^{1/3}\eta$. The appropriate boundary conditions are $f(0)=0$, $f'(0)=1$ since one plots $[u(x,y)]/[u(x,0)]$, $f'(\gamma)=0$ where γ is to be determined as the termination of the spreading jet, or $f''(0)=0$ if an initial value problem is preferred. The problem appears to be overdetermined, but it is not. The reduced differential equation that is obtained from the more general transformation is precisely the same one that was found by Tollmien.

The specific mathematical analysis that was originally employed by Tollmien and that has been religiously copied by textbook writers ever since is bizarre and dumbfounding to beginning students, particularly so, since straightforward and standard methods that are described in elementary books are adequate. This possibility of a straightforward integration may be implied by the absence of the Tollmien equation from the Kamke[3] collection of solvable ordinary differential equations since he includes some of the equations from fluid dynamics; nevertheless, a formal demonstration of the elementary solvability can settle the question and the curiosity of students. An integration by inspection leads to

$$ff' = (f'')^2 + k_1$$

and the initial conditions show quite fortuitously that $k_1 = 0$. The difficulty of the problem would have been enhanced enormously if k_1 did not vanish. Now, let $f' = p$ in accordance with the accepted method for equations when the independent variable does not appear explicitly, and the equation becomes $fp = [p\mathrm{d}p/\mathrm{d}f]^2$ or $p = \mathrm{d}f/\mathrm{d}\xi = [f^{3/2} + k_2]^{2/3}$. The initial conditions show once again that $k_2 = 1$, and separation of the variables yields

$$\xi + k_3 = \int \frac{\mathrm{d}f}{[f^{3/2}+1]^{2/3}}$$

The quadrature is nasty, but a sequence of substitutions,

$$f \to q^2,\ (q^3+1) \to r^3,\ r^3 \to 1/s,\ s-1 \to t^3$$

yields

$$\xi + k_3 = 2\int \frac{t\mathrm{d}t}{t^3+1}$$

$$= 2\left[1/6\ ln\ \frac{1-t+t^2}{(1+t)^2} + \sqrt{3}/3\tan^{-1}\frac{(2t-1)\sqrt{3}}{3}\right]$$

that could be easily performed by partial fractions or that can be found in all elementary tables of integrals, e.g. Dwight.[4] If one notes that $t(0)=0$, then $k_3 = -\pi - 3/9$ and

$$9\xi = \pi\sqrt{3} + ln\ \frac{1-t+t^2}{(1+t)^2} + 6\sqrt{3}\tan^{-1}\frac{(2t-1)\sqrt{3}}{3}$$

where

$$t^3 = \frac{f^{3/2}}{1+f^{3/2}} \ \cdots$$

It is easily seen that $t=\infty$ corresponds to $f'(\gamma)=0$ or to $f=-1$ and that $\gamma = 4\pi\sqrt{3}/9$. In addition to its simplicity the other principal advantage of this derivation is the inclusion of another adjustable parameter m in addition to c so that the radius of curvature of u/u_o could be increased near the plane of symmetry of the jet or so that the similar variable yx^β could be adjusted in a least squares sense or in some other fashion to compensate for the systematic deviations (positive near the plane of symmetry and negative far from it) that are observed in the experimental profiles and that are seen in the linear plots for y/x.[2]

Before leaving the Tollmien equation it may be interesting to note a direct application of a Galerkin approximation. The profiles measured by Albertson et al.[5] suggest that the shape of the plot of the horizontal component of the plane turbulent jet is Gaussian; hence, it seems appropriate on physical grounds to assume that $f'(\xi) = \exp(-\alpha\xi^2)$. To use this assumption one puts the Tollmien equation into the form

$$(f')^3 = 2f'f''f''' - (f'')^3$$

and one notes that the initial conditions are satisfied provided that

$$f(\xi) = \int_0^\xi \exp(-\alpha\xi^2)\,d\xi.$$

The Galerkin method gives an approximation for α by solving

$$\int_0^\xi (f')^4 d\xi = 2\int_0^\infty (f')^2 f'' f''' d\xi - \int_0^\infty f' (f'')^3 d\xi$$

or

$$\int_0^\infty \exp(-4\alpha\xi^2)d\xi = 8\alpha^2 \int_0^\infty \xi \exp$$

$$\times(-4\alpha\xi^2)d\xi - 8\alpha^3 \int_0^\infty \xi^3 \exp(-4\alpha\xi^2)d\xi$$

or

$$\sqrt{\pi}/4\sqrt{\alpha} = \alpha - \tfrac{1}{4}d$$

or

$$\alpha = (\pi/9)^{1/3}$$

Therefore, one finally has that $u/u_0 = \exp[-\xi^2(\pi/9)^{1/3}]$ with ξ containing both m and c that can be selected to fit "best" the "best" data.

References

[1] Tollmien, W., "Berechnung turbulenter Austreitungsvorgänge," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 6, 1926, pp. 468-478.

[2] Abramovich, G. N., *The Theory of Turbulent Jets*, M.I.T. Press, Cambridge, 1963, Chaps. 1 and 2, particularly §§1.3-1.6 and 2.1-2.5, pp. 3-125.

[3] Kamke, E., *Differentialgleichugen, Lösungsmethoden und Lösungen*, 3rd ed., Chelsea, New York, 1948, pp. 542-600.

[4] Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, 4th ed., Macmillan, New York, 1961, Integral 165.11, p. 40.

[5] Albertson, M. L., Dai, Y. B., Jensen, R. A., and Rouse, H., "Diffusion of Submerged Jets," *Proceedings ASCE*, Vol. 74, 1948, pp. 1571-1596.

SERT II Spacecraft Thruster Restart, 1974

William R. Kerslake* and Robert C. Finke†
NASA Lewis Research Center, Cleveland, Ohio

THE SERT II (Space Electric Rocket Test II) spacecraft was launched into a 100-km, polar, sun-synchronous orbit in February 1970 with a goal of demonstrating long-term operation of an ion thruster in space.[1] Thruster 1 was operated for 5½ months and then thruster 2 was operated for 3 months at 6.3 mlb thrust, 4200 sec specific impulse with 850 w input power. Thruster operation was terminated in each case by a high-voltage short due to an eroded web of the accelerator grid.

By 1973 the orbit had precessed such that the sun angle was oblique and only marginal power was available. To obtain more solar power the spacecraft was tipped over and spin stabilized such that the solar array was in a plane normal to the sun. A notable result was that during the first test of thruster 2 the high-voltage short was found to be cleared following the spin maneuver.

During the 1974 test period thruster 2 was restarted 19 times and run to thrust levels limited only by the available solar power. The high-voltage short remains in thruster 1, but its cathodes were started 12 times to show restart capability. The propellant feed systems, power processors, and spacecraft ancillary equipment were demonstrated to be functional after 4½ years in space.[2] In addition to thruster tests, a neutralizer cathode was operated separately to demonstrate that the potential level of a spacecraft could be controlled by the neutralizer alone.

Continued precession of the orbit will bring a continuous sun orbit in 1980 and the possibility of continuous thruster operation. Presently, shadow portions of the orbit prevent more than fractional (<1 hr) orbit periods of solar power operation.

This Note presents the highlights of the data taken during 1974. For a more complete discussion of the data and the 15-cm diameter mercury electron bombardment thruster, the reader is referred to the conference preprint[3] and earlier SERT II references.[4-6]

Thruster Operation

Thruster turn-on and operate commands were limited to real time (~20 min) periods while maintaining spacecraft contact over a ground station. Thruster 2 beam-on time varied from a few seconds to 40 min (two ground stations in sequence). Comparison data for thruster operation in 1974 with

Table 1 Performance of flight thruster 2

	30% beam		80% beam		Telemetry uncertainty (rss)
Year	1970	1974	1970	1974	
Day	2/11	9/10	2/11	9/11	
Restart number	10	198	10	200	
Main vaporizer heater	[a]1.63 [a]1.51	1.70 1.77	1.70 1.70	1.85 1.95	±0.07 V ±0.08 A
Main cathode heater	7.9 1.54	8.7 1.57	8.3 1.54	8.7 1.57	±0.35 V ±0.05 A
Main discharge	42.2 0.7	42.2 0.6	41.5 1.2	41.4 1.1	±0.2 V ±0.05 A
Beam voltage Beam current	[d]3490 [d]0.088	[d]2960 [d]0.083	[d]3160 [d]0.203	[d]2630 [d]0.198	±65 V ±0.005 A
Accelerator grid	[d]-1730 1.1	[d]-1480 0.9	[d]-1640 1.4	[d]-1330 1.4	±50 V ±0.1 mA
Neutralizer heater	[a]6.6 [a]2.0	8.1 2.3	[a]6.4 [a]1.9	7.5 2.2	±0.25 V ±0.05 A
Neutralizer keeper	27.8 [d]0.215	27.8 [d]0.175	[c]24.0 [d]0.206	[c]27.8 [d]0.167	±0.7 V ±0.004 A
Spacecraft voltage	-17	-8	-17	(e)	±2 V
Neutralizer emission	0.087	0.080	0.201	0.195	±0.006 A
Main cathode keeper	20.4 [b]0.282	20.0 [b]0.272	13.9 [b]0.283	13.1 [b]0.272	±0.5 V [b]±0.003 A
Solar array voltage	68	59	63	52	±1.0 V

[a] Heater power lower due to higher thermal background.
[b] Estimated value.
[c] Values due to different set points.
[d] Difference in values due to different solar array voltage input to power processor.
[e] Data unavailable.

Submitted March 12, 1975; presented as Paper 75-365 at the AIAA 11th Electric Propulsion Conference, New Orleans, La., March 19-21, 1975; revision received August 4, 1975.

Index categories: Electric and Advanced Space Propulsion; Spacecraft Flight Testing.

*Technical Consultant, Electric Propulsion Branch. Associate Fellow AIAA.

†Chief, Electric Propulsion Branch. Member AIAA.